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Relations between Homotopy and Homology II

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§ 1. Introduction

After the publication of my last paper, "Relations between homotopy and homology I" [1], I had an opportunity of reading H. Hopf's papers [2], [3]. In those papers I found that both his method and results were somewhat similar to mine. Let us compare the geometrical part, the grouptheoretical part being excluded. Hopf only thought of "Homotopie-ränder" and dealt with the subgroup Π_0 defined by the notion of free homotopy, while I dealt with three homomorphisms, namely, homotopy boundary, homotopy relativisation and covering, as a part of the exact homomorphism sequence of homotopy groups. For the lack of the idea of free homotopy in my method, I could not deal with the low dimensional case. The difference of methods leads to different results. Taking Hopf's idea into consideration, we can see the difference between Hopf's group $\mathfrak{G}^n$ and my "simple group" Θ^n, and its geometrical nature; moreover the homology theory of a complex will be found to be reduced, in a sense, completely into the homotopy theory.

§ 2. Hopf's free homotopy group

Let $\{\alpha_i\}$ be the generators of the q-dimensional homotopy group $\pi_q(R)$ of a locally contractible topological space R and $\{\xi_j\}$ be the generators of the fundamental group $\pi_1(R)$. The Whitehead's product $\xi_j \cdot \alpha_i$ is also an element of $\pi_q(R)$. We denote by Γ_q the normal subgroup generated by all the elements $\{\xi_j \cdot \alpha_i\}$. The free homotopy group $\tilde{\pi}_q(R)$ is defined as the factor group $\pi_q(R)/\Gamma_q(R)$, i. e. if we add new defining relations $\{\xi_j \cdot \alpha_i = 1\}$ to the relations of π_q, then we get the group $\tilde{\pi}_q$. We denote by u the natural homomorphism from π_q to $\tilde{\pi}_q$. It is easily seen from the definition that for any elements $\alpha \in \pi_q$, $\xi \in \pi_1$, $\xi \cdot \alpha = 1$ in $\tilde{\pi}_q$. If R is q-simple in the sense of Eilenberg [4], then it is clearly $\pi_q = \tilde{\pi}_q$.

Sometimes we take a relative operator domain $\pi_1(L)$, where L is a closed connected subset of R. Replacing $\pi_1(R)$ with $\pi_1(L)$ we get a normal subgroup $\Gamma_q' \subset \pi_q(R)$ as before and factor group $\tilde{\pi}_q'(R)=\pi_q(R)/\Gamma_q'$. We say this group $\tilde{\pi}_q'(R)$ as L-free homotopy group. We can define similarly the free relative homotopy group $\tilde{\pi}_q(R, L)$ as the factor group

$$\pi_q(R, L) \ / \ \Gamma_q(R, L),$$

where $\Gamma_q(R, L)$ is a normal subgroup of $\pi_q(R, L)$ which is generated by all the elements of the Whitehead's product $\{\xi_j \cdot \alpha_i\}$, $\xi_j \in \pi_1(L)$, $\alpha_i \in \pi_q(R, L)$.

Now we can define three homomorphisms, $\tilde{i}$, $\tilde{r}$ and $\tilde{\partial}$ with respect to the free homotopy groups, which correspond respectively to the homomorphisms i, r and ∂ of the homotopy groups, as follows.

1) Injection $\tilde{i}$: Let $\tilde{\alpha}$ be an element of $\tilde{\pi}_q(L)$, and α be an element of $u^{-1}(\tilde{\alpha}) \subset \pi_q(L)$, then we define $\tilde{i}(\tilde{\alpha})$ as the element $ui(\alpha) \in \tilde{\pi}_q'(R)$. We can see easily from the operator homomorphisms of the homotopy injection that this mapping $\tilde{i}$ is uniquely determined. Writing in operator symbol it is also $\tilde{i}u=ui$ for every element of $\pi_q(L)$. Clearly $\tilde{\pi}_1(L)$ is abelian, but $\tilde{\pi}_1'(R)$ is not necessarily abelian, for the operator domain is only $\pi_1(L)$ and not $\pi_1(R)$. If we take $\pi_1(R)$ as the domain, or the injection $i(\pi_1(L))$ is a mapping onto $\pi_1(R)$, then the free homotopy group $\tilde{\pi}_1'(R)$ is abelian.

2) Relativisation $\tilde{r}$: We define a mapping $\tilde{r}$ from $\tilde{\pi}_q'(R)$ into $\tilde{\pi}_q(R, L)$ such that $\tilde{r}u=ur$ in operator symbol.

3) Free homotopy boundary $\tilde{\partial}$: Similarly a mapping $\tilde{\partial}$ from $\tilde{\pi}_q(R, L)$ into $\tilde{\pi}_{q-1}(L)$ is defined such that $\tilde{\partial}u=u\partial$. It is easily verified that $\tilde{r}$ and $\tilde{\partial}$ are uniquely determined.

We get also the following homomorphism sequence

$$\tilde{\pi}_q(L) \xrightarrow{\tilde{i}} \tilde{\pi}_q'(R) \xrightarrow{\tilde{r}} \tilde{\pi}_q(R, L) \xrightarrow{\tilde{\partial}} \tilde{\pi}_{q-1}(L) \longrightarrow \ldots\ldots,$$

which is not necessarily exact. The image groups are included respectively in the kernels of the subsequent homomorphisms.

Now we shall state some remarks on the free homotopy groups which are easily verified from definition.

1) $\tilde{\pi}_2(R, L)$ is abelian. For, let $\tilde{\alpha}$, $\tilde{\beta}$ be any two elements of $\tilde{\pi}_2(R, L)$, and $u(\alpha)=\tilde{\alpha}$, $u(\beta)=\tilde{\beta}$. Clearly the homotopy boundary $\partial\alpha$ is an element of $\pi_1(L)$. From the J. H. C. Whitehead's relations $\alpha\beta\alpha^{-1}=\beta^{\partial\alpha}$, we can conclude that $\alpha\beta\alpha^{-1}\Gamma_2(R, L)=\beta^{x}\Gamma_2(R, L)=\beta\Gamma_2(R, L)$, i. e. $\tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1}=\tilde{\beta}$ in $\tilde{\pi}_2(R, L)$.

2) Let R be a simply connected two dimensional complex K^2 and K^1 be the 1-section of K^2. From the exact homomorphism seqence of homotopy groups we see easily that $\pi_2(K^2, K^1)$ is decomposed into the direct product of $\pi_2(K^2)$ and $\pi_1(K^1)$, i. e.

$$\pi_2(K^2, K^1) \approx \pi_2(K^2) \times \pi_1(K^1).$$

If we denote the commutator subgroup of $\pi_2(K^2, K^1)$ by $\mathfrak{c}$, then

$$\pi_2(K^2, K^1)/\mathfrak{c} \approx \tilde{\pi}_2(K^2, K^1),$$

and $\tilde{\pi}_2(K^2, K^1) \approx \pi_2(K^2) \times \tilde{\pi}_1(K^1)$.

§ 3. Principal Theorems

Let K be a connected and locally finite polyhedron and K^n be its n-section. The following Theorem I for the case $n>3$ is already stated in my paper [I], but in this paper I shall prove it with slightly different method for any dimension $n\,(>1)$.

Theorem I. *If K is simply connected, then the free homotopy group $\tilde{\pi}_2(K^2, K^1)$ is isomorphic with the 2-dimensional chain group of K with integer coefficients.*

Similarly we can prove the following

Theorem I$'$. *If K is simply connected, then the homotopy group $\pi_n(K^n, K^{n-1})$ $(n\geq 3)$ is isomorphic with the n-dimensional chain group of K with integer coefficients.*

For a general complex K we can prove easily the following Theorem II which is already used in my paper [I] for the case $n>2$.

Theorem II. *The relative free homotopy group $\tilde{\pi}_n(K^n, K^{n-1})$ is isomorphic with the n-dimensional chain group of K with integer coefficients.*

Let $\bar{K}$ be the universal covering complex of K and u be the covering mapping from $\bar{K}$ onto K. Then u defines also a mapping from the n-dimensional chain group $L^n(\bar{K}, I)$ of $\bar{K}$ onto the n-dimensional chain group $L^n(K, I)$ of K. This homomorphic mapping is denoted also by u as in [I]. Then we get the following relation

Theorem III. *The homomophic mapping u of $L^n(\bar{K}, I)$ onto $L^n(K, I)$ corresponds exactly to the free homotopic mapping of $\pi_n(K^n, K^{n-1})$ onto $\tilde{\pi}_n(K^n, K^{n-1})$, that is, there holds the relation*

$$u\,\varphi = \tilde{\varphi}\,u,$$

$$\begin{array}{ccc}
\pi_n(K^n, K^{n-1}) & \xrightarrow{u} & \tilde{\pi}_n(K^n, K^{n-1}) \\
\downarrow \varphi & & \downarrow \tilde{\varphi} \\
L^n(\bar{K}, I) & \xrightarrow{u} & L^n(K, I),
\end{array}$$

where the mapping φ and $\tilde{\varphi}$ are those defined in Theorem I and II.

From these theorems we get an important diagram, which gives many interesting results as in [I],

$$\begin{array}{ccccc}
\pi_n(K^n, K^{n-1}) & \xrightarrow{\partial_t} & \pi_{n-1}(K^{n-1}) & \xrightarrow{r} & \pi_{n-1}(K^{n-1}, K^{n-2}) \\
\downarrow u & & & & \\
\tilde{\pi}_n(K^n, K^{n-1}) & \xrightarrow{\tilde{\partial}_t} & \tilde{\pi}_{n-1}(K^{n-1}) & \xrightarrow{\tilde{r}} & \tilde{\pi}_{n-1}(K^{n-1}, K^{n-2}) \\
\downarrow \tilde{\varphi} & & & & \downarrow \tilde{\varphi} \\
L^n(K, I) & & \xrightarrow{\partial_l} & & L^{n-1}(K, I),
\end{array}$$

where $\pi_2(K^2, K^1)$ has to be replaced by the group $\pi_2(K^2, K^1)/\mathfrak{c} \approx \tilde{\pi}_2(K^2, K^1)$. In any dimension $\pi_n(K^n, K^{n-1})$ can be considered as the direct sum of the group $\Gamma_n(K^n, K^{n-1})$ and the group $\tilde{\pi}_n(K^n, K^{n-1})$ as in [I].

Hopf's "Homotopierände" is the image of the homomorphic mapping $\tilde{\partial}_t\tilde{\varphi}^{-1}$ of $L^n(K, I)$ in this diagram. We can see more clearly from this figure the difference between the spherical cycles and the Hopf's "Henkel" cycles. The details of various kinds of cycles will be seen in § 5.

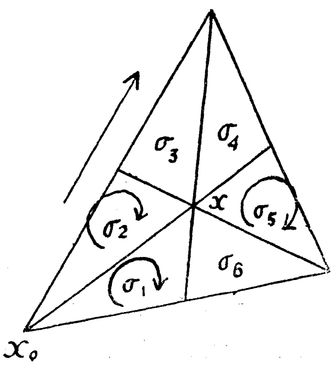
§ 4. Proofs of Theorem I, II and III.

We prove first some lemmas which state probably somewhat elementary properties.

Let T^n be a closed oriented n-simplex, $T^n_{(i)}$ be the i-th baricentric derived of T^n and x_0 be a vertex of the boundary of T^n. We denote by σ^n_i $(i=1, \dots, (n+1)!)$ the oriented simplexes of $T^n_{(1)}$ such that the homological boundary $\partial_l(\sum\sigma^n_i)$ is equal to the subdivided complex $(\partial_l T^n)_{(1)}$. Let σ^n_i $(i=1,2, \dots, n!)$ be incident with x_0. We denote by x the center of T^n.

The oriented n-cube E^n is represented by n coordinates x_i $(i=1, 2, \dots, n)$ with $0 \leq x_i \leq 1$. E^n may be divided into two parts E^n_0, E^n_1 such that E^n_0 is the part with $0 \leq x_1 \leq 1/2$ and E^n_1 the part with $1/2 \leq x_1 \leq 1$.

Lemma 1. *Let f_i $(i=1, 2)$ be the mappings of E^2 on σ^2_i $(i=1,2)$ such that f_i are orientation preserving homeomorphisms and $f_i(0)=x_0$, where 0 is the origin of E^2. Let f_i $(i=3, 4, 5, 6)$ be the mappings of E^2 on $(x_0x) \cup \sigma^2_i$ $(i=3, 4, 5, 6)$ such that $f_i(E^2_0)=(x_0, x)$, $f_i(0)=x_0$, and $f_i(E^2_1)$ are orientation preserving homeomorphisms. Let β_i be the element of $\pi_2(T^2_{(1)},$*



$(T^2_{(1)})'$, x_0) *which is given by the mapping* f_i $(i=1,\dots,6)$. *Then the element* $\beta_2\beta_3\dots\beta_6\beta_1$ *is equal to the element* β *of* $\pi_2(T^2_{(1)}, (T^2_{(1)})^1, x_0)$ *which is given by such a mapping* f *that* $f(E^2)=T^2_{(1)}$ *is an orientation preserving homeomorphism and* $f(0)=x_0$.

Lemma 2. *Let* σ^2_j *be an oriented 2-simplex of* $T^2_{(i)}$ *and* w_j *be a suitable arc which connects a fixed vertex* x_j *of* σ^2_j *with* x_0. *The elements* β_i, β *are defined similarly as in Lemma 1. Then there holds the relation*

$$\beta=\Pi_{j=1}^{(3!)^i}\beta_j,$$

where the product is to be taken through all the simplexes of $T^2_{(i)}$ *with an appropriate order.*

The proof of Lemma 2 can be performed by repeating the process of Lemma 1 i-times. The analogous theorem for the case of dimension $n>2$ can be proved more easily neglecting the consideration of the arc (x_0x), for $(T^n_{(i)})^{n-1}$ is simply connected, and hence $(n$-1)-simple in the sense of S. Eilenberg.

Lemma 2′. *Let* β, β_j *be elements of* $\pi_n(T^n_{(i)}, (T^n_{(i)})^{n-1})$ $(n>2)$ *which are given by the homeomorphic mappings of* E^n *onto* T^n, $\sigma^n_j\in T^n_{(i)}$ *respectively. Then there holds the relation*

$$\beta=\sum\beta_i.$$

Now we proceed to consider the relative homotopy group $\pi_n(K^n, K^{n-1})$ of a general complex K. Let y_0 be a fixed vertex of K. To any n-simplex σ^n_j of K we correspond a fixed arc $u_i=y_0y_i$, which connects the point y_0 with a point y_i of σ^n_i.

Let α_i be the element of $\pi_n(K^n, K^{n-1})$, which is defined by the mapping f_i of E^n onto $(y_0y_i)\bigcup\sigma^n_i$ such that

$$f_i(E^n_0)=u_i,\ f_i(E^n_1)=\sigma^n_i,$$

where $f_i(E^n_1)$ is an orientation preserving homeomorphism.

Lemma 3. *The elements* $\{\alpha_i^{\xi_j}\mid i=1,\dots,p;\ j=1,\dots,q\}$ *are generators of* $\pi_2(K^2, K^1)$, *where* i *runs over all the 2-simplexes and* j *over all the elements of* $\pi_1(K^1)$.

Proof. Let α be an element of $\pi_2(K^2, K^1)$ which is given by the following mapping f:

$$f(E^2)\subset K^2,\ f(\partial_t E^2)\subset K^1,\ f(x_0)=y_0.$$

We approximate f by a suitable simplicial mapping φ from the k-times baricentric subdivided element $E^2_{(k)}$ into K^2. If we denote the homotopy between f and φ by $h_t\,(0\leq t\leq 1)$ such that $h_0\equiv f$, $h_1\equiv\varphi$, we can clearly choose such a deformation so that we have $h_t(\partial E^2)\subset K^1$ and $h_t(x_0)=y_0$. Therefore the element of $\pi_2(K^2, K^1)$, which is defined by φ, is α.

Notation being as in Lemma 2, the element β_i of $\pi_2(\overset{2}{E}_{(i)}, (\overset{2}{E}_{(i)}))$ goes by φ onto an element $(\alpha_{l(i)}^{\xi_{m(i)}})^{\pm 1}$ and β onto α. From the homomorphism of the relative homotopy group by simplicial mappings, it follows that

$$\alpha=\varphi(\beta)=\Pi\varphi(\beta_i)=\Pi\,(\alpha_{l(i)}^{\xi_{m(i)}})^{\pm 1}$$

in $\pi_2(K^2, K^1)$; this proves the lemma.

Lemma 3′. *Let* $\{\sigma_i \mid i=1,\ldots,p\}$ *be the n-simplexes of* K, *and* ξ_j *be the elements of* $\pi_1(K)$, *then the elements* $\{\alpha_i^{\xi_j}\}$ *constitute the generators of* $\pi_n(K^n, K^{n-1})$ $(n>2)$.

The proof is similar to that of lemma 3. We have only to use lemma 2′ instead of lemma 2.

Corollary. $\{\alpha_i \mid i=1,\ldots,p\}$ *are generators of* $\tilde{\pi}_n(K^n, K^{n-1})\,(n>1)$.

Proof of Theorem I, I′.

It is sufficient to show that $\tilde{\pi}_n(K^n, K^{n-1})$ with generators $\{\alpha_i\}$ is free abelian, for K being simply connected, $\pi_n(K^n, K^{n-1})$ is isomorphic with $\tilde{\pi}_n(K^n, K^{n-1})$ $(n>2)$.

If any relation $\sum m_i\,\alpha_i=0$ exists in $\tilde{\pi}_2(K^2, K^1)$, then there exists an element c of $c\subset\pi_2(K^2, K^1)$ such that

$$c\,\Pi\,(\alpha_i)^{m_i}=1$$

in $\pi_2(K^2, K^1)$. If a mapping f gives the element $c\,\Pi\,(\alpha_i)^{m_i}$, then there exists a homotopy $f_t\,(0\leq t\leq 1)$ such that

$$f_0\equiv f$$
$$f_1(E^2)\subset K^1.$$

Now we identify the subset $K^2-\overset{2}{\sigma_i}$ of K^2 to a point y_0. Then K goes to a sphere S^2. We denote this identification by ω. The mapping ωf of E^2 onto S^2 gives an element of $\pi_2(S^2)$, and f_t shows that $\omega\,(c\,\Pi\,(\alpha_i)^{m_i})=0$ in $\pi_2(S^2)$.

Mapping ω transforms the element $\alpha_j\,(j\neq i)$ of $\pi_2(K^2, K^1)$ into 0 of $\pi_2(S^2)$, and α_i into ± 1 of $\pi_2(S^2)$. Therefore ω transforms the element $\sum m_i\,\alpha_i$ into $\pm m_i$. But this value of $\pi_2(S^2)$ must be 0, which shows Theorem I. The proof of Theorem I′ is similar.

Let $\bar{K}$ be the universal covering complex of K, then the n-simplexes of K are represented by $\{\xi_j, \overset{n}{\sigma_i}\}$, where ξ_j, $\overset{n}{\sigma_i}$ are those defined in Lemma 3. From Theorem I, I′ and Hurewicz's isomorphism we can easily see that $\pi_n(\bar{K}^n, \bar{K}^{n-1}) \approx \pi_n(K^n, K^{n-1})$ $(n>2)$ is free abelian. In the case of dimension 2 $\pi_2(K^2, K^1)/c \approx \tilde{\pi}_2(\bar{K}^2, \bar{K}^1)$ is also free abelian.

Proof of Theorem II, II′. If we examine the content of the proof of Theorem I, I′, we may obtain the proof of Theorem II, II′.

Remark: From theorem II we can see immediately that $\tilde{\pi}_n(K^n, K^{n-1})$ is a subgroup of $\pi_n(K^n, K^{n-1})$ and that

$$\pi_n(K^n, K^{n-1}) \approx \Gamma_n(K^n, K^{n-1}) + \tilde{\pi}_n(K^n, K^{n-1}).$$

Proof of Theorem III.

The correspondence between the group $\tilde{\pi}_n(K^n, K^{n-1})$ and $L^n(K, I)$ is given from Theorem II by the following correspondence $\tilde{\varphi}$ of their generators:

$$\tilde{\varphi}: \ \alpha_i \longrightarrow \sigma_i.$$

The isomorphism between $\pi_n(K^n, K^{n-1})$ and $L^n(\bar{K}, I)$ is given similarly by the correspondence

$$\varphi: \ \overset{\xi_i}{\alpha_i} \longrightarrow \xi_i \overset{n}{\sigma_i}.$$

The homomorphism u from $\pi_n(K^n, K^{n-1})$ onto $\tilde{\pi}_n(K^n, K^{n-1})$ is given, as above mentioned, by the correspondence

$$u: \ \overset{\xi_i}{\alpha_i} \longrightarrow \alpha_i,$$

and the covering mapping from $\bar{K}$ onto K is clearly defined by the correspondence

$$u: \ \xi_j\sigma_i \longrightarrow \sigma_i$$

From these correspondences Theorem III is clear.

In concluding this paragraph we state the main theorem about homotopy and homology.

From theorem II we may consider the group $\tilde{\pi}_n(K^n, K^{n-1})$ $(n\geq 2)$ as the chain group $L^n(K, I)$ itself. Then the homology boundary ∂_l defines a homomorphism from $\tilde{\pi}_n(K^n, K^{n-1})$ into $\tilde{\pi}_{n-1}(K^{n-1}, K^{n-2})$. $\tilde{\pi}_n(K^n, K^{n-1})$, considered as a subgroup of $\pi_n(K^n, K^{n-1})$ is transformed by the composite operation $ur\partial_t$ into $\tilde{\pi}_{n-1}(K^{n-1}, K^{n-2})$. Then there holds the following theorem.

Theorem 4. *In operator symbol* $\partial_l = ur_t\,\partial_t$, *or more precisely*

$$\partial_l = ur_t\,\partial_t\,u^{-1}$$

The proof is just the same as in [I]. The homomorphisms u and ∂_t are defined by homotopy only, and the relativisation r_t is defined by homotopy with respect to the dimension. Therefore we may insist that the homology theory is reducible to the homotopy theory.

Now we can deduce from this theorem various kinds of cycles and various aspects of homological properties of a complex.

§ 5. Applications

Using the relations $\tilde{\partial}_t u = u\partial_t$, $\tilde{r}u = ur$, we get the following Corollary. In operator symbol $\partial_t = \tilde{r}\tilde{\partial}_t$. A chain c^n, satisfying the relation $\partial_t u^{-1}(c^n)=0$, is named "spherical" by Eilenberg. A chain c^n, satisfying the relation $r\partial_t u^{-1}(c^n)=0$, is named by the author "simple". Hopf named the cycle c^n, which satisfies the relation $\tilde{\partial}_t(c^n)=0$, i. e. $u\partial_t u^{-1}(c^n)=0$ "Henkel".

The Henkel cycles constitute a subgroup $\bar{\mathfrak{P}}^n$ of Z^n and the factor group $\bar{\mathfrak{P}}^n/B^n$ is denoted by $\mathfrak{P}^n$.

The simple cycles constitute a subgroup $\bar{\Theta}^n$ of Z^n, and the factor group $\bar{\Theta}^n/B^n$ is denoted by Θ^n.

Then we get following diagram about subgroups of H^n:

$$\sum^n - \Theta^n \wedge \mathfrak{P}^n \begin{smallmatrix} \Theta^n \\ < \quad > \\ \mathfrak{P}^n \end{smallmatrix} \Theta^n \vee \mathfrak{P}^n - \mathfrak{Q}^n - H^n$$

where $\sum^n$ is the spherical group, and $\Theta^n \vee \mathfrak{P}^n$ is the subgroup of H^n which is generated by Θ^n and $\mathfrak{P}^n$, and $\mathfrak{Q}^n$ is defined, in terms of λ, μ, ν, by

$$\mathfrak{Q}^n = \bar{\mathfrak{Q}}^n/B^n,\ \mathfrak{G}^n = u\,\partial_t^{-1}(\lambda_{n-1}(K^{n-1}) \wedge (\nu_{n-1}(K^{n-1}) \vee \Gamma_{n-1}(K^{n-1}))).$$

To see the geometrical properties of Θ^n, $\mathfrak{P}^n$, $\mathfrak{Q}^n$ and $H^n/\mathfrak{Q}^n$, it is convenient to represent them by the groups λ, μ, ν. We can prove the following relations:

i) $\lambda_{n-1}(K^{n-1}) \wedge \Gamma_{n-1}(K^{n-1})/\partial_t(\Gamma^n(K^n, K^{n-1})) \approx \mathfrak{P}^n/\sum^n$,

ii) $\lambda_{n-1}(K^{n-1}) \wedge \nu_{n-1}(K^{n-1})/\partial_t(\Gamma^n(K^n, K^{n-1})) \wedge \nu_{n-1}(K^{n-1}) \approx \Theta^n/\sum^n$,

iii) $\lambda_{n-1}(K^{n-1}) \wedge \Gamma_{n-1}(K^{n-1}) \wedge \nu_{n-1}(K^{n-1})/\partial_t(\Gamma_n(K^n, K^{n-1}))$
$\wedge \nu_{n-1}(K^{n-1}) \approx \mathfrak{P}^n \wedge \Theta^n/\sum^n$,

iv) $(\lambda_{n-1}(K^{n-1}) \wedge \nu_{n-1}(K^{n-1})) \vee (\lambda_{n-1}(K^{n-1}) \wedge \Gamma_{n-1}(K^{n-1}))/\partial_t$
$(\Gamma_n(K^n, K^{n-1})) \approx \mathfrak{P}^n \vee \Theta^n/\sum^n$,

v) $\lambda_{n-1}(K^{n-1}) \wedge \nu_{n-1}(K^{n-1})/\Gamma_{n-1}(K^{n-1}) \wedge \nu_{n-1}(K^{n-1}) \wedge \lambda_{n-1}(K^{n-1})$
$\approx \mathfrak{P}^n \vee \Theta^n/\mathfrak{P}^n$,

vi) $\lambda_{n-1}(K^{n-1}) \wedge \Gamma_{n-1}(K^{n-1})/(\lambda_{n-1}(K^{n-1}) \wedge \Gamma_{n-1}(K^{n-1})) \wedge (\nu_{n-1}(K^{n-1})$
$\vee \partial_t(\Gamma_n(K^n, K^{n-1}))) \approx \mathfrak{P}^n \vee \Theta^n/\Theta^n$,

vii) $\lambda_{n-1}(K^{n-1}) \wedge (\nu_{n-1}(K^{n-1}) \vee \Gamma_{n-1}(K^{n-1}))/\partial_t(\Gamma_n(K^n, K^{n-1})) \approx \mathfrak{Q}^n/\sum^n$,

viii) $r(\lambda_{n-1}(K^{n-1})) \wedge \Gamma_{n-1}(K^{n-1}, K^{n-2})/r(\Gamma_{n-1}(K^{n-1}) \wedge r(\lambda_{n-1}(K^{n-1}))$
$\approx H^n/\mathfrak{Q}^n$.

Relation (i) shows that the homotopy boundary of an element of $\mathfrak{P}^n$ is included in $\Gamma_{n-1}(K^{n-1})$, i.e. of the form

$$\sum \xi \cdot \beta$$

where $\beta \in \pi_{n-1}(K^{n-1})$ and $\xi \in \pi_1(K^{n-1})$. We can see easily from this fact the elements of $\mathfrak{P}^n$ are represented by mappings of Hopf's "Henkel"-manifold.

Relation (ii) shows that the homotopy boundary of an element of Θ^n is included in $\nu_{n-1}(K^{n-1})$, i.e. the dimension of the image of the boundary is at most n-2.

Now we define a pseudo-sphere as follows: From an m-sphere S^m we cut off k elements E^m_k, then the boundary of the rest is composed of k $(m-1)$-spheres S^m_k. We identify them respectively to k cycles $Z^{p(k)}_k$ with dimensions $p(k) < m-1$. By this construction we get a pseudo-sphere.

It is easy to prove that any element of Θ^n is represented by a continuous mapping from a suitable pseudo-sphere into K^n, and conversely any such mapping will give an element of Θ^n. For example a fibre bundle over the base space S^r and with the fibre S^s $(r>1, s>1, r+s=n)$ is a pseudo-sphere and its continuous mapping into K^n gives an element α of Θ^n. In this case if $\pi_{n-1}(S^r)=0$, $\pi_{n-1}(S^s)=0$, then the element α gives rise to an element of $\sum^n$.

Now it is easy to understand what mappings represent the elements of $\mathfrak{Q}^n$.

The geometrical aspect of the elements of $H^n/\mathfrak{Q}^n$ is as follows:

viii) $H^n/\mathfrak{Q}^n \approx r(\lambda_{n-1}(K^{n-1})) \wedge \Gamma_{n-1}(K^{n-1}, K^{n-2})/r(\Gamma_{n-1}(K^{n-1})) \wedge r(\lambda_{n-1}(K^{n-1}))$

If we neglect the consequences of $\mathfrak{Q}^n$, an element of H^n, i.e. a cycle is given by such a mapping that the boundary $\partial_t(E^n)$ is transformed into $\alpha\xi\alpha^{-1}$, where $\alpha \in \pi_{n-1}(K^{n-1}, K^{n-2})$, $\xi \in \pi_1(K^{n-2})$. Therefore we decompose the boundary S^{n-1} into $2q$ pieces of elements $E^{n-1}_{i,1}$, $E^{n-1}_{i,2}$ $(i=1, \dots, q)$, and map each $E^{n-1}_{i,1}$ and $E^{n-1}_{i,2}$ into a non-spherical element α_i. If this mapping f of $\partial_i(E^n)$ can be extended to that of E^n, then the mapping $f(E^n)$ gives an element of H^n, which is not an element of $\mathfrak{Q}^n$.

From these considerations we can conclude properties of groups $\sum^n$, Θ^n, $\mathfrak{P}^n$, $H^n/\Theta^n \cup \mathfrak{P}^n$ for special complexes and dimensions. For instances:

$\Theta^3/\sum^3=0$; if $\pi_i(K)=0$ $(i=2, \dots, n-2)$, then $\Theta^n \approx \sum^n$ and $\Theta^{n-1} \approx \sum^n$; if $\pi_{n-1}(K)=0$, then $\mathfrak{P}^n \approx \sum^n$; and so forth.

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